REPORT 13

FINITE-FREQUENCY RESOLUTION LIMITS OF TRAVELTIME TOMOGRAPHY FOR SMOOTHLY VARYING VELOCITY MODELS

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ABSTRACT

The Rytov approximation, which expresses phase residuals as an explicit function of the slowness perturbations, is also a Generalized Radon Transform (GRT). Using Beylkin’s formalism, we derive the corresponding inverse GRT to give the slowness model as an explicit function of the phase residuals. This expression is used to deduce the resolution limits of traveltime tomograms as a function of source frequency and source-receiver geometry. Its validity is restricted to arbitrary models with smooth variations in velocity, where the velocity variations must be at least 3 times longer than the characteristic source wavelength. The formula shows that the slowness resolution limits of an anomaly can be computed by estimating the intersection area of the wavepaths that visit it. Using this procedure, resolution limits are obtained for several types of data: controlled source data in a crosswell experiment, data from a refraction experiment, and earthquake data using the reference velocity model of the whole earth.

INTRODUCTION
Current 3-D models of the whole earth are primarily based on velocity images calculated by tomographic inversion of earthquake travel times. Such models have revolutionized our understanding of the earth’s origins, convection cells, and tectonic mechanisms. However, tomographic images are limited in accuracy because the high-frequency assumption of ray theory conflicts with the inherent low frequency of the observed teleseismic arrivals. This conflict leads to an inaccurate estimate of the model errors because the finite-frequency effects are taken into account.

Exploration seismologists also use ray-based tomographic methods to determine subsurface velocity structures, and so their velocity models are also limited by finite-frequency effects in the data. With few exceptions, tomographers generally ignore the wave-interference effects that can significantly limit the resolution of their tomographic images.

To correct this deficiency, Woodward and Rocca (1988), Woodward (1989 and 1992), and Luo and Schuster (1990, 1991) developed inversion methods that accounted for finite-frequency effects in the traveltime data under the, respectively, Rytov and Born approximations. Instead of backprojecting traveltime residuals along raypaths, Woodward and Rocca’s formulation backprojected phase residuals along Rytov wavepaths in the space-frequency domain while Luo and Schuster backprojected traveltime residuals along Born wavepaths in the space-time domain. In either case, the finite-frequency effects of wave propagation were partly accounted for and led to more accurate tomograms with low-frequency data. Here, the region of a first-Fresnel zone or wavepath is defined by the following condition (see Kravtsov and Orlov, 1980):

\[ |\tau (r', r_s) + \tau (r', r_g) - \tau (r_s, r_g)| \leq \frac{T}{2} \]  

(13.1)

where \( T \) is the period, \( \tau (r'r) \) is the traveltime for waves to propagate from \( r \) to \( r' \), and the point \( r' \) belongs to the first-Fresnel zone for the source-receiver pair \( (r_s, r_g) \) if only if it satisfies equation (13.1).

The problem with the wave-equation methods is that they are still too expensive to routinely implement, particularly with earthquake-tomography studies where the teleseismic waves have propagated over thousands of kilometers. A cheaper, but less effective, means of accounting for wave interference effects in traveltime tomograms is to derive resolution limits based on wavepath effects, and to incorporate these into estimates of model variances for ray-based tomograms. Until now, no such formulae have been derived for arbitrary velocity models.

Now, we derive such resolution limits for traveltime tomograms computed for arbitrary earth models with smooth variations in the velocity. The starting point is to relate the phase to the model by the Rytov approximation, recognize the resulting equation as a Generalized Radon Transform (GRT), and use Beylkin’s (1985) formalism to derive the inverse GRT. The inverse GRT explicitly represents the slowness model as a function of phase data so that resolution limits of the reconstructed slowness model can be obtained. These limits are valid in the asymptotic high-frequency
limit, but are still useful for finite-frequency phenomena. To paraphrase Bleistein (1984): "...the results of the asymptotic analysis are usually meaningful when the typical wavelength is (in practice 3 times or more) shorter than the typical dimension in the problem." Velocity images obtained by earthquake traveltime tomography, reflection traveltime tomography, or refraction tomography can now be assessed for their limits of resolution based on the finite-frequency effects of the data.

We divide this paper into three parts. The first section presents the derivation of the inverse GRT formula that explicitly relates the reconstructed model to the phase data. From this formula we derive the resolution limits of the slowness model as a function of source frequency and source-receiver coverage. The second section presents some numerical examples where the resolution limits are computed for a crosswell experiment, a surface refraction experiment, and whole earth tomography. Finally, the conclusions are presented. To enhance the readability of this paper, the mathematical derivations are mostly placed in the appendices.

**METHODOLOGY**

The diffraction-slice theorem (Wu and Toksoz, 1987) for a homogeneous velocity model was extended by Sheng (1998) to the case of wavepath traveltime tomography in an inhomogeneous velocity model. In his result, the estimated slowness or object function $O^{\text{est}}(r)$ reconstructed from phase residuals can be approximated as:

$$O^{\text{est}}(r) = \frac{1}{(2\pi)^n} \int_{\Omega(r)} e^{-ik \cdot r} \hat{O}(k) dk,$$  \hspace{1cm} (13.2)

where the object function is given as $O(r) = \frac{1}{V^2(r)} - \frac{1}{V_0^2(r)}$; $V_0(r)$ denotes the background actual velocity; $n$ is the model dimension; $\hat{O}(k)$ denotes the Fourier transform of $O^{\text{est}}(r)$; and $k$ is the wavenumber vector given by

$$k = \omega \left( \nabla \tau (r, r_g) + \nabla \tau (r, r_s) \right).$$  \hspace{1cm} (13.3)

Here, $\omega$ represents source frequency; $\nabla \tau (r, r_g)$ and $\nabla \tau (r, r_s)$ are the gradients of the traveltimes for waves to propagate from the image point at $r$ to the geophone $r_g$, and from the source at $r_s$ to $r$, respectively; and both $\tau (r, r_g)$ and $\tau (r, r_s)$ satisfy the eikonal equation and can be calculated by conventional ray tracing (see Appendix A) for a given velocity model and source-receiver geometry. In equation (13.2), the integration domain $\Omega(r)$ in the Fourier domain determines the spatial resolution of the reconstructed object function $O^{\text{est}}(r)$ and controls what can be recovered (Beylkin, 1985). As an example, the maximum value of $k_x$ or $k_z$ in the integration limits of
equation (13.2) determines the smallest resolvable features in the horizontal (i.e., $\Delta x$) and vertical (i.e., $\Delta z$) directions.

Therefore the resolution limits of the traveltime tomogram at some point $\mathbf{r}$ can be estimated as:

$$\Delta x_i (\mathbf{r}) = \frac{2T}{\max_{T, \eta_{sg}} \left| \nabla \tau (\mathbf{r}, \mathbf{r}_g) + \nabla \tau (\mathbf{r}, \mathbf{r}_s) \right|} \quad (13.4)$$

where $\Delta x_i$ is the resolution limit along the coordinate direction $x_i$. For finite frequencies, the region in the vicinity of the raypath that mostly influences the recorded traveltime for a given source-receiver pair is usually denoted as a wavepath (Rocca and Woodward, 1998) or a first-Fresnel zone (Červený and Soares, 1992), so $\eta_{sg} (\mathbf{r})$ represents the source-receiver pairs for which point $\mathbf{r}$ is within the first-Fresnel zone of the corresponding wavepaths.

As an example, reflection energy that emanates from a scatterer at $\mathbf{r}$ will arrive at every receiver, thus $\eta_{sg} (\mathbf{r})$ in equation (13.4) includes all possible source-receiver pairs. This idea is illustrated in Figure 13.1a. In contrast, the source-receiver pairs for transmission tomography are restricted to those wavepaths which have a central ray that connects $\mathbf{r}_s$ and $\mathbf{r}_g$ by the Fermat transmission ray (see Figure 13.1b), and also where the associated wavepaths intersect $\mathbf{r}$. These source-receiver pairs are many fewer than those for reflection tomography and this results in a poorer resolution for transmission traveltime tomography.

**NUMERICAL EXAMPLES**

Some numerical examples will now be computed for estimating slowness resolution limits for different types of synthetic seismic data: traveltime data from a seismic-crosswell experiment, traveltimes from a surface-refraction experiment, and earthquake traveltimes for whole-earth tomography.

**Crosswell Experiment**

Assume the crosswell configuration shown in Figures 13.1a and 13.1b, where some localized slowness perturbation is embedded in a homogeneous model with a velocity $c$. The wells are offset by a distance $X$, and the vertical length of the source and geophone wells is $L$. For a specified source at $\mathbf{r}_s = (0, z_s)$, and geophone at $\mathbf{r}_g = (X, z_g)$, and image point at $\mathbf{r}_0 = (x, z)$, equation (13.3) and Appendix A says that the wavenumber of the estimated slowness model for a wavepath that intersects the point $\mathbf{r}_0$ is given by:

$$k_x (x, z) = \frac{\omega}{c} \left( \frac{x}{\sqrt{x^2 + (z - z_s)^2}} + \frac{x - X}{\sqrt{(x - X)^2 + (z - z_g)^2}} \right) \quad (13.5)$$
and
\[ k_z(x, z) = \frac{\omega}{c} \left( \frac{z - z_s}{\sqrt{x^2 + (z - z_s)^2}} + \frac{z - z_g}{\sqrt{(x - X)^2 + (z - z_g)^2}} \right). \]  

(13.6)

The above formulas can now be used to give the resolution limits for both reflection and transmission tomography.

**Reflection traveltime tomography.** The resolution limits for reflection traveltime tomography can be obtained by identifying the allowable source ((0, z_s)) and receiver ((X, z_g)) coordinates that maximize the wavenumbers in equation (13.5) and (13.6). Thus, the maximum resolvable wavenumber of a scatterer at \( r_0 \) is given as:

\[
\max_{r_s \in R_s, r_g \in R_g} |k_x(x, z)| = \frac{\omega}{c} \left[ 1 - \min \left( \frac{|X - x|}{\sqrt{(x - X)^2 + (z \pm L/2)^2}}, \frac{|x|}{\sqrt{x^2 + (z \pm L/2)^2}} \right) \right],
\]

and

\[
\max_{r_s \in R_s, r_g \in R_g} |k_z(x, z)| = \frac{\omega}{c} \max \left[ \frac{|L/2 \pm z|}{\sqrt{x^2 + (L/2 \pm z)^2}} + \frac{|L/2 \pm z|}{\sqrt{(x - X)^2 + (L/2 \pm z)^2}} \right].
\]

(13.7)

(13.8)

For an image point in the middle of the model \( r_0 = (X/2, 0) \), when \( X \gg L \), the above equations reduce to:

\[
\Delta x \approx \frac{4\lambda X^2}{L^2},
\]

(13.9)

and

\[
\Delta z \approx \frac{\lambda X}{L},
\]

(13.10)

which are the same as the migration spatial-resolution limits for crosswell migration derived by Schuster (1996) in the far-field approximation.

**Transmission traveltime tomography.** For a fixed source at \( r_s = (0, z_s) \) and a scatterer at the center \( r_0 = \left( \frac{X}{2}, 0 \right) \) the maximum resolvable wavenumber can be obtained by identifying the \((r_s, r_g)\) pair such that the boundary of its associated first-Fresnel zone intersects the scatterer at \( r_0 \). As mentioned earlier, the allowable source-receiver pairs for transmission tomography are far fewer than for reflection tomography. Thus, Appendix B shows that the slowness wavenumbers reconstructed by transmission-traveltime tomography are maximum for the following formulas:

\[
|k_x| \approx \frac{\omega}{c} \frac{4|z_s|\sqrt{\lambda}}{(X^2 + 4z_s^2)^{3/4}},
\]

(13.11)

and

\[
|k_z| \approx \frac{\omega}{c} \frac{2X\sqrt{\lambda}}{(X^2 + 4z_s^2)^{3/4}},
\]

(13.12)
For reflection tomography, any source-receiver pair \((r_s, r_g)\) has a wavepath which includes the specified scatterer at \(r_0\); b). For transmission tomography, the source-receiver pairs are many fewer. For a fixed source at \(r_s\), the associated geophones are restricted to be those between \(r_{g1}\) and \(r_{g2}\), where point \(r_0\) is on the boundary of the Fresnel zones for source-receiver pairs \((r_s, r_{g1})\) and \((r_s, r_{g2})\); c). The intersection region of the Fresnel zones also defines the spatial resolution limits at \(r_0\). The blackened area, which is the intersection of three wavepath Fresnel zones, defines the spatial resolution limits at \(r_0\) for these three wavepaths; d). The football-shaped intersection of the first-Fresnel zones for a continuous distribution of sources and receivers. Here, \(X = 200\) (\(m\)), \(L = 400\) (\(m\)), velocity and the frequency are taken to be \(3000\) (\(m/s\)) and \(300\) (\(Hz\)), respectively. The size of the intersection region is: \(\Delta x = 72\) (\(m\)), and \(\Delta z = 44.7\) (\(m\)), which is consistent with the spatial resolution limits predicted by equations (13.14) and (13.15).
and we assume $X \gg \frac{\lambda}{2}$.

The spatial resolution limits can be found by choosing the source that maximizes the above wavenumbers $|k_x|$ and $|k_z|$, i.e., for $X > \frac{\sqrt{2}}{2} L$,

$$\Delta x \approx \frac{(X^2 + L^2)^{\frac{3}{2}}}{L} \sqrt{\lambda};$$  \hspace{1cm} (13.13)

for $X \leq \frac{\sqrt{2}}{2} L$,

$$\Delta x \approx \frac{3\sqrt{\lambda X}}{\sqrt{12}};$$  \hspace{1cm} (13.14)

and

$$\Delta z \approx \sqrt{\lambda X},$$  \hspace{1cm} (13.15)

which are similar to the resolution limits $\Delta x = \frac{\sqrt{3}(X^2)}{L} \sqrt{\lambda}$ and $\Delta z = \sqrt{\frac{\lambda X}{2}}$ in Schuster (1996) for traveltime tomography in the far-field approximation.

Wavepath traveltome tomography backprojects the traveltime residual to be within the first-Fresnel zone of the wavepath, so the location of the slowness anomaly is unresolved within the intersection of the Fresnel zones; thus the intersection of the Fresnel zones defines the spatial resolution limits of reconstructed model. For this example, when $X \gg \frac{1}{2}$ the intersection region of the Fresnel zones can be calculated by geometric construction to be,

for $X > \frac{\sqrt{2}}{2} L$,

$$\Delta x \approx \frac{(X^2 + L^2)^{\frac{3}{2}}}{L} \sqrt{\lambda};$$  \hspace{1cm} (13.16)

for $X \leq \frac{\sqrt{2}}{2} L$,

$$\Delta x \approx \frac{3\sqrt{\lambda X}}{\sqrt{12}};$$  \hspace{1cm} (13.17)

and

$$\Delta z \approx \sqrt{\lambda X}.$$  \hspace{1cm} (13.18)

The formulas above are the same as those obtained by maximizing the wavenumbers (equations (13.14) - (13.15)), but this method of geometric construction is needed for arbitrary velocity distributions. We will now use the geometric method to estimate the spatial resolution limits for traveltime inversion of refraction and earthquake data.

**Refraction traveltome tomography under the far-field approximation**

A two-layer velocity model will be assumed and refraction traveltimes will be used to reconstruct the slowness model. According to equation (13.4), the horizontal $\Delta x$
and vertical $\Delta z$ resolution limits of a refraction tomogram are then given by:

$$\Delta x \approx \lambda_2,$$

and,

$$\Delta z \approx \frac{\lambda_1}{4} \cos(\theta),$$

where $\lambda_2$ is the wavelength in the lower medium, and $\lambda_1$ is the wavelength in the upper medium. Equation (13.19) can be derived by noting that the refraction ray along the interface defines the maximum value of $k_x = \frac{2\pi}{\lambda_2}$. The vertical resolution $\Delta z$ in equation (13.20) can be derived by noting that a scatterer can be lifted from the refraction point to the point $B$ in Figure 13.2b and still contribute refraction energy to the arrival within $T/2$ of the initial onset. Equation (13.20) also defines the maximum value of $k_z$ for the refraction rays in Figure 13.2.

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Figure 13.2: Refraction raypaths associated with horizontal resolution (top figure) and vertical resolution (lower figure). Note that as the critical angle $\theta$ goes to zero the vertical resolution approaches $\frac{\lambda_1}{4}$, that of reflection migration. Here, the trace is depicted above the receiver at $R$ and the source is located at $S$. 
Global earthquake tomography

Whole-earth velocity tomograms can be obtained by inverting the first-arrival travel-times picked from teleseismic records. Such tomograms are limited in resolution by the finite-frequency effects of low-frequency waves propagating through the earth. To understand these limits we calculated the wavepaths for 1 Hz teleseismic P-waves.

The Preliminary Reference Earth Model or PREM (Dziewonski et. al., 1981) was taken to be the background velocity model, and traveltimes were computed by an eikonal-equation solver. These traveltimes were then used to numerically calculate the Fresnel zones defined by equation (13.1), and the intersections of the Fresnel zones were used to define the spatial-resolution limits of the tomogram. The sources and geophones are distributed uniformly around a great circle at a spatial interval of 1 degree for every source (or receiver); the dominant frequency is set to be 1 Hz; and the grid point interval is taken to be 2 km.

Figure 13.3 shows the wavepaths superimposed onto the PREM, where only first-arrival traveltimes are considered and we focus on the region no deeper than 1500 km. Figure 13.4 shows the intersection region of the first-Fresnel zones at the depths of 100 km, 300 km, 400 km, and 800 km. The result shows that the spatial resolution generally gets worse at deeper depths. At the depths between 100 km and 300 km, the spatial resolution (horizontal or vertical) is about 64 km (or ±32 km away from the point under consideration), between 400 km and 900 km, it is about 150 km (or ±75 km).

SUMMARY

We have derived the inverse GRT for the Rytov equation, which yields the reconstructed slowness as an explicit function of phase residuals. This formula also provides a practical means for estimating the limits of model resolution due to wave-interference effects. It is valid for arbitrary earth models with smoothly varying velocities having variations longer than three times the source wavelength. For the crosswell example, resolution formulae are derived that are in agreement with formulae derived under the far-field approximation. For the earthquake tomography example, a procedure is defined that allows for the computation of the wavepath resolution limits of the earthquake tomograms for given source-receiver geometries, image point locations, and source frequencies. And for the surface-refraction problem, simple formulas are given which yield vertical-and horizontal-resolution limits of a refraction tomogram associated with a layered-earth model.

One of the implications of this work is that the reliability estimates for global tomograms should take into account wavepath resolution limits. It may be necessary to revise some interpretations of whole-earth tomograms due to the neglect of wave interference effects. Another implication is that the different resolution limits along each coordinate axis give rise to oblong artifacts (e. g., Figure 13.1d) in the recon-
Figure 13.3: The transmission wavepaths that intersect a single point at a depth of 800 km. The sources and receivers are distributed uniformly around a great circle of the earth at a spatial interval of 1 degree for every source (or receiver), the dominant frequency is set to be 1 Hz, and the grid point interval is taken to be 2 km. The units of both the $X$- and $Z$-coordinates are in km, and the units of velocity are km/s. The image point is located at $(800, 6371)$, and the fat rays are the wavepaths.
Figure 13.4: The approximate intersection regions of the Fresnel zones at different depths. The image points under consideration are at the center of the quadrilaterals which represent the intersection regions of the first-Fresnel zones. The size of the quadrilaterals gives the spatial resolution limits of an earthquake tomogram, where wave interference effects are partly taken into account. The horizontal width gives the horizontal resolution $\Delta x$ and the vertical height gives the vertical resolution $\Delta z$. The sources and the geophones are uniformly distributed around a great circle of the earth for 1 degree every source (or geophone). a) Depth=100 km, $\Delta x = 64$ km, $\Delta z = 48$ km; b) Depth= 300 km, $\Delta x = 64$ km, $\Delta z = 96$ km; c) Depth= 400 km, $\Delta x = 96$ km, $\Delta z = 144$ km; d) Depth= 800 km, $\Delta x = 164$ km, $\Delta z = 148$ km.
structured model. A corrective procedure should be devised, otherwise these artifacts will be present in the tomogram and can lead to erroneous interpretations.

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REFERENCES


APPENDIX A: INVERSE GRT

Under the Rytov and high-frequency approximations, the traveltime residual \( \Delta t (r_s, r_g) \) can be linearly related with the object function \( O (r) \) as,

\[
\Delta t (r_s, r_g) = -(\frac{1}{2})^{(n-1)/2} \int_{\Omega} O (r) A (r, r_g) e^{i\omega \phi (r, r_g)} \, dr, \tag{13.21}
\]

where \( n \) represents the model dimension, \( O (r) = \frac{1}{V_0 (r)} - \frac{1}{V (r)} \); \( V_0 (r) \) denotes the background actual velocity, \( A (r, r_g) = a (r, r_s) a (r, r_g) / a (r_s, r_g) \) and \( \phi (r, r_g) = \tau (r, r_s) + \tau (r, r_g) - \tau (r_s, r_g) \), in which \( \tau (r, r') \) and \( a (r, r') \) satisfies the eikonal equation and the transport equation, respectively. According to Beylkin (1985), equation (13.21) can be related to a causal Generalized Radon Transform. For a fixed source \( r_s \), the object function can be expressed as,

\[
O^{est} (r) = \frac{1}{(2\pi)^{n}} \int_{\Omega} \int_{\partial X} \frac{\Delta t (r_s, r_g) e^{-i\omega \phi (r, r_g)} h (r, r_g) A (r, r_g)}{A (r, r_g)} \, dr_g \omega^{(n-1)/2} \, d\omega, \tag{13.22}
\]

where \( h (r, r_g) \) is Beylkin’s determinant (Beylkin, 1985). Substituting equation (13.21) into equation (13.22) yields:

\[
O^{est} (r) = \frac{1}{(2\pi)^{n}} \int_{\Omega} \int_{\partial X} e^{i\omega \phi (r, r_g)} A (r, r_g) h (r, r_g) O (r') \, dr'dr_g \omega^{n-1} \, d\omega. \tag{13.23}
\]

Following Beylkin’s approach, equation (13.23) can be approximated as,

\[
O^{est} (r) = \frac{1}{(2\pi)^{n}} \int_{\Omega} \int_{\partial X} e^{i\omega \nabla_r \phi (r, r_g) \cdot (r' - r)} h (r, r_g) O (r') \, dr'dr_g \omega^{n-1} \, d\omega. \tag{13.24}
\]

Changing variables of integration from \( \omega, r_g \) to \( k \), where

\[
k = \omega \nabla_r \phi (r', r_g), \tag{13.25}
\]

we get,

\[
O^{est} (r) = \frac{1}{(2\pi)^{n}} \int_{\Omega (r)} \int_{\partial X} e^{i\mathbf{k} \cdot (r' - r)} O (r') \, dr'd\mathbf{k}, \tag{13.26}
\]

where \( \Omega (r) \) is the image of \( R (\omega) \times \partial X (r) \) under the change of variables in equation (13.25), and \( \partial X (r) \). It follows from equation (13.26) that

\[
O^{est} (r) = \frac{1}{(2\pi)^{n}} \int_{\Omega (r)} e^{-i\mathbf{k} \cdot r} O (k) \, dk, \tag{13.27}
\]
where $\hat{O}(k)$ is the Fourier transform of the function $O(r)$. For the formulas and the derivation of this Beylkin-Rytov wavepath traveltime tomography refer to Sheng (1998).

**APPENDIX B: CROSSWELL RESOLUTION LIMITS**

In this appendix we derive the crosswell resolution limits using equation (13.4). For a crosswell geometry with a constant velocity $c$ (shown in Figure 13.1b), the maximum resolvable wavenumber for a scatterer at the centre $r_0 = \left(\frac{X}{2}, 0\right)$ can be obtained for a fixed source $r_s = (0, z_s)$ by identifying the source-receiver pair $(r_s, r_g)$ such that $r_0$ is on the boundary of the Fresnel zone for this pair. At some geophone $r_g = (X, z_g)$, we have,

$$\sqrt{\left(\frac{X}{2}\right)^2 + z_s^2} + \sqrt{\left(\frac{X}{2}\right)^2 + z_g^2} = \sqrt{X^2 + (z_s - z_g)} + \frac{\lambda}{2}. \quad (13.28)$$

When $X \gg \frac{\lambda}{2}$, we obtain,

$$|z_g + z_s| \approx \frac{|z_s|\sqrt{X}(X^2 + 4z_s^2)^{\frac{3}{2}}}{X}. \quad (13.29)$$

Using a differential approximation to equations (13.5) and (13.6), the wavenumbers can be expressed as,

$$|k_x| \approx \frac{\omega 4X|z_s| |z_g + z_s|}{c \left(X^2 + 4z_s^2\right)^{\frac{3}{2}}}, \quad (13.30)$$

and,

$$|k_z| \approx \frac{\omega 2X^2|z_g + z_s|}{c \left(X^2 + 4z_s^2\right)^{\frac{3}{2}}}. \quad (13.31)$$

Substituting equations (13.29) into equation (13.30) and (13.31), equation (13.11) and (13.12) are derived. For $z_s$ within $\left[-\frac{L}{2}, \frac{L}{2}\right]$, the maximum of $|k_x|$ is obtained when $z_s = \pm \frac{L}{2}$, for $X > \frac{\sqrt{2}}{2}L$, and $z_s = \frac{\sqrt{2}}{2}X$, for $X \leq \frac{\sqrt{2}}{2}L$; the maximum of $|k_z|$ is obtained when $z_s = 0$. Hence, we get the resolution estimates in equations (13.13) through (13.15).

Assume there is some ray from a source at $r_s = (0, z_s)$ passing through $r_0 = \left(\frac{X}{2}, 0\right)$ to a geophone at $r_g = (X, -z_s)$. If $r_1 = (x, 0)$ and $r_2 = \left(\frac{X}{2}, z\right)$ are on the boundary of the Fresnel zone, the size of the intersection of the Fresnel zones can be estimated by calculating the minimum value of $2|x - \frac{X}{2}|$ and $2|z|$, where $x$ and $z$ satisfy,

$$\sqrt{\left(\frac{X}{2}\right)^2 + (z - z_s)^2} + \sqrt{\left(\frac{X}{2}\right)^2 + (z + z_s)^2} = \sqrt{X^2 + 4z_s^2} + \frac{\lambda}{2}. \quad (13.32)$$
and,
\[
\sqrt{x^2 + \frac{z^2}{s^2}} + \sqrt{(x-X)^2 + \frac{z^2}{s^2}} = \sqrt{X^2 + 4z^2} + \frac{\lambda}{2}.
\] (13.33)

Solving these two equations for \(x\) and \(z\), when \(X \gg \frac{\lambda}{2}\) and \(L \gg \frac{\lambda}{2}\), we have,
\[
2|x - \frac{X}{2}| \approx \frac{\sqrt{\lambda} (X^2 + 4z^2)^{\frac{3}{2}}}{2zs},
\] (13.34)

and,
\[
2|z| \approx \frac{\sqrt{\lambda} (X^2 + 4z^2)^{\frac{3}{2}}}{X}.
\] (13.35)

Minimizing \(2|x - \frac{X}{2}|\) and \(2|z|\) for \(z_s\) within \([-\frac{L}{2}, \frac{L}{2}]\), yields equations (13.16) through (13.18).