ELECTROMAGNETIC VELOCITY INVERSION USING
TWO-DIMENSIONAL MAXWELL’S EQUATIONS

by

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ABSTRACT

The wave equation traveltime inversion (WT) method is adapted to the reconstruction of a dielectric distribution from traveltime radar data. A gradient optimization algorithm is used and the gradient function is computed from finite-difference solutions to the 2-D Maxwell's equations. The key advantage of the radar WT method over conventional ray tracing radar tomography is that it accounts for scattering and diffusion effects in the data and works well in both resistive and moderately conductive rocks. This technique is successfully applied to both synthetic and real radar data. Comparisons with a ray tracing (RT) tomography scheme show that the radar WT method is more robust and accurate than the RT method when rock conductivity is larger than 0.002 S/m. The WT and RT methods are about equally effective when conductivity is less than or equal to 0.001 S/m. The disadvantage of the WT scheme is that it generally demands an order of magnitude more computational time than the RT method.
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INTRODUCTION

High-frequency electromagnetic waves will propagate explorationally significant distances with little attenuation through certain low conductivity materials such as granites, limestones, coals, and dry salt (Cook, 1975). This fact has led to the development of many electromagnetic techniques for determining electrical and geological properties of the earth (Lytle, 1974; Lytle et al., 1976). One of these techniques is ground probing radar, which is a fast, safe, and economical means for estimating crosshole or near surface earth properties. Using phase delay and attenuation losses in the radar signal, radar systems can be used to map salt dome structures (Stewart and Unterberger, 1976), to delineate rock fracture patterns (Lodha et al., 1991), and to calculate the thickness of coal seams and glacier ice (Coon et al., 1981; Jezek, 1985; Hammond and Sprenke, 1991). Other uses of radar include mapping stratigraphic features and the extent of stress-relief cracks in potash mines (Annan et al., 1988), and discovering buried cities in Yemen, ancient footpaths in Costa Rica, and roads and archeological sites in Chaco Canyon, New Mexico (New York Times, March 10, 1992).

There are two types of conventional methods for determining the dielectric distribution. One is to invert for the dielectric constant $\varepsilon_r$ from phase shifts in the radargrams by assuming a linear relation between the phase shift $\Delta \Phi$ and the wave propagation slowness $s = \sqrt{\mu_0 \varepsilon}$ for a specific frequency $\omega$, i.e., $\Delta \Phi = \omega s$ (Cook, 1975; Lytle et al., 1976). In this case $\varepsilon_r = \varepsilon / \varepsilon_0$, where $\varepsilon$ is the permittivity of the rock, $\varepsilon_0$ is that of free space, and $\mu_0$ is the permeability of free space. Another method is ray tracing inversion from time delays of radargrams (Lager
and Lytle, 1977; Lytle, 1980; Frank and Balanis, 1989). Both methods are based on the same assumption that the rock conductivity does not significantly affect the first arrival traveltime (phase shift) of the electromagnetic waves. For a less conductive rock where diffusion effects are negligible, ray tracing or phase shift methods may be used to accurately reconstruct the dielectric distribution. For higher conductive media, where conductivity effects on traveltime measurements are significant, ray tracing methods are inadequate and may give incorrect results. Because many field sites are characterized by non-negligible conductivity values, there is a compelling need to extend the radar tomography method to the moderate conductivity regime (.001 S/m < σ ≤ .01 S/m).

The purpose of this thesis is to overcome the conductivity problems in ray tracing traveltime inversion (RT) by adapting the wave equation traveltime inversion (WT) method of Luo and Schuster (1991) to Maxwell's equations. Instead of using ray tracing, the WT method uses finite-difference solutions to Maxwell's equations to both model and invert the radar traveltime data. Unlike ray tracing, the WT method accounts for both the electromagnetic wave propagation and diffusion effects in the data. Hence, a more accurate reconstruction of the dielectric distribution is possible. The geometry of interest is that of a radar crosshole experiment and the goal will be to invert for the dielectric parameter distribution ε (or electromagnetic velocity $c = \sqrt{\frac{\varepsilon}{\mu}}$) from radar traveltime data.

The first section of this paper presents the radar WT derivation for Maxwell's equations. This provides a velocity updating formula associated with a gradient optimization algorithm. The second section gives a physical interpretation and numerical demonstration of the WT method. The third section presents test results for inversion of both synthetic and real radar crosshole data. This demonstrates that the WT method is more accurate and reliable than an RT
method when diffusion effects are significant. The last section contains our conclusion and a discussion about future work.
RADAR WT INVERSION

This section presents the WT algorithm for inverting the electromagnetic velocity distribution from first arrival radar travel times. The strategy consists of 4 steps: (1) define a traveltime misfit function, (2) establish a connective function that links the traveltime residuals to the electric field radargrams, (3) derive the gradient of the misfit function, and (4) use an iterative gradient formula to update the velocity distribution.

For an electric current line source in a 2-D medium, Maxwell’s equations in the time domain (Ward and Hohmann, 1988) are given by

\[
\begin{align*}
\frac{\partial E_y}{\partial z} &= \mu \frac{\partial H_x}{\partial t}, \\
\frac{\partial E_y}{\partial x} &= -\mu \frac{\partial H_z}{\partial t}, \\
\frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon} \left[ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y - j_y \right],
\end{align*}
\] (1)

where \( j_y \) is the \( y \)-component of the electric current density, \( E_y \) is the electric field strength along the \( y \) direction, and \( (H_x, H_z) \) denotes, respectively, the \( x \) and \( z \) components of the magnetic field intensity. Parameters \( \mu, \varepsilon, \) and \( \sigma \) represent the magnetic permeability, electric permittivity or dielectric distribution, and electrical conductivity, respectively.

Since the magnetic permeability \( \mu \) is nearly a constant in most geologic environments (Keller, 1988), system (1) is equivalent to the equation

\[
\nabla^2 E_y - \mu \sigma \frac{\partial E_y}{\partial t} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = \mu \frac{\partial j_y}{\partial t},
\] (2)
where \( c = 1/\sqrt{\mu \varepsilon} \) is the radar velocity. Equation (2) will be used to derive the gradient formula of the radar WT method.

To numerically solve for \( E_y \), we apply an explicit 2-4 staggered grid finite-difference scheme to system (1) (Yee, 1966; Levander, 1988). The details of this finite-difference algorithm are described in Appendix A.

To invert for the dielectric distribution \( \varepsilon(x) \) or radar wave velocity \( c(x) = 1/\sqrt{\mu \varepsilon(x)} \), we will assume that: (1) the medium is two-dimensional, linear, and isotropic, (2) the transient line source emulates the major traveltime characteristics of 3-D wave propagation, (3) the magnetic permeability \( \mu \) is that of free space \( (\mu = \mu_0) \), and the conductivity distribution \( \sigma(x) \) is known within some tolerance (the specific tolerance limits will be determined numerically), and (4) the transmitter-receiver array has the crosshole geometry shown in Figure 1.

The following steps outline the WT inversion algorithm for the reconstruction of the radar velocity.

(1) Form a traveltime residual misfit function \( e \)

\[
e = \frac{1}{2} \sum_s \sum_r \Delta T^2(x_r, x_s),
\]

where the summations are over the source \( x_s \) and receiver \( x_r \) indices, and \( \Delta T(x_r, x_s) \) is the first arrival traveltime residual or difference between the observed and calculated first arrival traveltimes in the electric field time records (i.e., radargrams). Following the notation from Luo and Schuster (1991), we will call \( \Delta T(x_r, x_s) \) the skeletalized data and the electric field \( E_y(x_r, t; x_s) \) the fundamental data that satisfies the fundamental governing equations, Maxwell’s equations.

(2) Find the gradient \( \gamma(x) \) of the traveltime misfit function with respect to
Figure 1. Crosshole radar geometry.
the velocity parameters

\[ \gamma(x) = \frac{\partial e}{\partial c(x)} = \sum_s \sum_r \Delta T(x_r, x_s) \frac{\partial \Delta T}{\partial c(x)}. \]  

(4)

(3) Compute \( \frac{\partial \Delta T}{\partial c(x)} \) from a connective function that links the skeletalized data \( \Delta T(x_r, x_s) \) to the fundamental data \( E_y(x_r, t; x_s) \). This connective function is defined by the cross-correlation between the observed electrogram \( d(x_r, t; x_s) \) and the calculated electrogram \( E_y(x_r, t; x_s) \), i.e.,

\[ f(x_r, \Delta T; x_s) = \int d(x_r, t+\Delta T; x_s) E_y(x_r, t; x_s) \, dt, \]  

(5a)

where we will assume that all but the first arrivals are muted out from the electrograms. The cross-correlation function \( f \) reaches a maximum at \( \Delta T = \Delta T^* \), where \( \Delta T^* \) is the time shift where the calculated electrograms best fit the shifted observed data. Here, \( \Delta T^* \) is the traveltime residual or difference between the first arrivals of the observed and calculated electrograms. Note that if the assumed velocity is the actual velocity then \( \Delta T^* = 0 \).

Differentiating equation (5a) we get

\[ \dot{f}(x_r, \Delta T^*; x_s) = \int \dot{d}(x_r, t+\Delta T^*; x_s) E_y(x_r, t; x_s) \, dt = 0, \]  

(5b)

where \( \dot{f} \) represents the derivative of \( f \) with respect to \( \Delta T \). We can consider \( \dot{f} \) in equation (5b) to be a function of \( \Delta T^* \) and \( c(x) \), where \( E_y(x_r, t; x_s) \) is an implicit function of \( c(x) \). Equation (5b) therefore acts as a constraint between \( c(x) \) and \( \Delta T^* \), i.e., \( \Delta T^* = \Delta T^*(c) \).

To find the dependence of \( \Delta T^* \) on \( c(x) \), we take the total derivative of (5b),

\[ df = \frac{\partial f}{\partial c} \delta c + \frac{\partial f}{\partial \Delta T^*} \delta \Delta T^* = 0, \]

and rearrange the above equation to get the Fréchet derivative,
\[
\frac{\partial \Delta T^*}{\partial c} = \lim_{\delta c \to 0} \frac{\Delta T^*}{\delta c} = - \frac{\frac{\partial f}{\partial c}}{\frac{\partial f}{\partial \Delta T^*}}. \tag{5c}
\]

Explicitly, \(\frac{\partial f}{\partial c}\) in the numerator of equation (5c) is calculated from equation (5b) to give

\[
\left. \frac{\partial f}{\partial c} \right|_{\Delta T^*} = \int \dot{d}(x_r, t; x_s) E_y(x_r, t; x_s) \frac{\partial E_y(x_r, t; x_s)}{\partial c(x)} dt,
\tag{6a}
\]

and \(\frac{\partial f}{\partial \Delta T^*}\) in the denominator of equation (5c) is given as

\[
\left. \frac{\partial f}{\partial \Delta T^*} \right|_{\Delta T^*} = \frac{\partial}{\partial \Delta T^*} \left[ \int \dot{d}(x_r, t \pm \Delta T^*; x_s) E_y(x_r, t \pm \Delta T^*; x_s) dt \right] \bigg|_{\Delta T^*},
\]

so that substituting \(t' = t + \Delta T\) into the above equation gives

\[
= \left. \frac{\partial}{\partial \Delta T^*} \int \dot{d}(x_r, t'; x_s) E_y(x_r, t' - \Delta T; x_s) dt' \right|_{\Delta T^*}
= - \int \dot{d}(x_r, t'; x_s) \dot{E}_y(x_r, t' - \Delta T; x_s) dt' \bigg|_{\Delta T^*}
= - \int \dot{d}(x_r, t + \Delta T^*; x_s) \dot{E}_y(x_r, t; x_s) dt
= - F, \tag{6b}
\]

where \(F\) is a constant for fixed \(x_r, x_s,\) and \(\Delta T^*\).

The representation for \(\frac{\partial E_y(x_r, t; x_s)}{\partial c(x)}\) in equation (6a) can be found by applying a small perturbation \(\delta c(x)\) to \(c(x)\), i.e., \(c'(x) = c(x) + \delta c(x)\), and then finding the associated perturbation \(\delta E_y(x)\) in the \(E_y(x)\) field, i.e., \(E'_y(x) = E_y(x) + \delta E_y(x)\). Substituting these perturbations into equation (2), and
invoking Green’s theorem (see Appendix B) we get

\[
\frac{\partial E_y(x_r, t; x_s)}{\partial c(x)} = -\frac{2}{c(x)^3} \int_0^t dt_0 \frac{\partial^2 E_y(x, t_0; x_s)}{\partial t_0^2} G(x_r, t; x, t_0). \tag{6c}
\]

where \(G(x_r, t; x, t_0)\) is the Green’s function associated with Maxwell’s equations.

Substituting equations (6a), (6b), and (6c) into (5c) and then substituting the resulting expression into equation (4) gives the formula for the gradient function \(\gamma(x)\)

\[
\gamma(x) = \frac{\partial e}{\partial c(x)}
\]

\[
= \frac{1}{c^3(x)} \sum_s \sum_r \int_0^\infty \delta T(x_r, t; x_s) \left[ \int_0^t dt_0 \ddot{E}_y(x, t; x_s) G(x_r, t; x, t_0) \right] dt.
\]

Interchanging the order of integrations we have

\[
\gamma(x) = \frac{1}{c^3(x)} \sum_s \int_0^t dt_0 \sum_r \int_0^\infty \delta T(x_r, t; x_s) G(x_r, t; x, t_0) dt_0 \]

\[
= \frac{1}{c^3(x)} \sum_s \int_0^t dt_0 \ddot{E}_y(x, t_0; x_s) E^b(x, t_0; x_s) dt_0, \tag{7}
\]

where the pseudo-residual \(\delta T\) is defined as

\[
\delta T(x_r, t; x_s) = -\mu \frac{\partial}{\partial t} \left[ \frac{2}{\mu F} d(x_r, t+\Delta T^*; x_s) \Delta T^*(x_r, x_s) \right], \tag{8a}
\]

and the weighted back-propagation term is given by

\[
E^b(x, t_0; x_s) = \int_0^\infty \sum_r \delta T(x_r, t; x_s) G(x_r, t; x, t_0) dt. \tag{8b}
\]

By reciprocity (Morse and Feshbach, 1953), \(G(x_r, t; x, t_0) = G(x, -t_0; x_r, -t)\)

so that \(E^b\) in equation (8b) becomes

\[
E^b(x, t_0; x_s) = \int_0^\infty \sum_r \delta T(x_r, t; x_s) G(x, -t_0; x_r, -t) dt. \tag{9}
\]
INTERPRETATION

This section provides a physical interpretation of the diffusion or conductivity effects on first arrival traveltimes. In addition, the gradient formula in equation (7) is interpreted in terms of back-projecting traveltime residuals along wavepaths.

Conductivity Effects On Traveltimes

Equation (2) is a damped wave equation. In a hard rock site where $\sigma$ is small and comparable to the value of $\varepsilon$, both the wave-like features and the diffusive behavior of the electric field are important. Consider, for simplicity, the whole space Green’s function for equation (2), which is defined as the solu-

$$\left( \nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \right) G_{dw}(x, t) = -\mu I \delta(x - x_s) \delta(t), \quad (11a)$$

in the time domain or

$$\left( \nabla^2 - i \mu \sigma \omega + \mu \varepsilon \omega^2 \right) G_{dw}(x, \omega) = -\mu I \delta(x - x_s), \quad (11b)$$

in the frequency domain, where $G_{dw}(x, t)$ is given by (Wolf, 1979)

$$G_{dw}(x, t) = \frac{\mu I}{2\pi} \frac{e^{-(\sigma/2\varepsilon)cosh\left[\frac{\sigma}{2\varepsilon}(t^2 - R^2/c^2)^{1/2}\right]}} {(t^2 - R^2/c^2)^{1/2}} H(t - R/c). \quad (12)$$

Here $R = |x - x_s|$ is the distance from the source position $x_s$ to a point $x$, $I$ is the electric current strength, $H(t - R/c)$ is the Heaviside step function, and $c = 1/\sqrt{\mu \varepsilon}$ is the electromagnetic wave velocity in a homogeneous whole space. Now consider the two extreme cases: the case where $\sigma = 0$ (wave propagation only) and the case where $\sigma \gg \varepsilon \omega$ (pure diffusion).
Case 1. $\sigma = 0$

For a highly resistive medium, the Green's function for the 2-D wave equation in equation (11a) is

$$G_w = \frac{\mu f}{2\pi} \frac{H(t - R/c)}{(t^2 - R^2/c^2)^{1/2}},$$

(13)

where the incident energy first arrives at the time $t = R/c$, i.e., the first arrival traveltime $t$ is a function of $c = 1/\sqrt{\mu\varepsilon}$. In this case, the first arrival traveltime can be correctly picked so that a ray tracing method may correctly reconstruct the velocity $c$ or dielectric distribution $\varepsilon$.

Case 2. $\sigma \gg \varepsilon \omega$

For a highly conductive medium ($\sigma \gg \varepsilon \omega$) the Green's function for equation (11a) becomes (Ward and Hohmann, 1984)

$$G_d = \frac{\mu f}{4\pi t} e^{-\mu \sigma R^2 / (4t)}.$$  

(14)

Taking the time derivative of $G_d$ and setting $\frac{\partial G_d}{\partial t} = 0$ shows that the largest amplitude in the signal arrives at the time $t = \mu \sigma R^2 / 4$; the signal that arrives at earlier times is exponentially small so that its traveltime cannot be correctly picked. In this situation, the picked traveltime is independent of the wave propagation velocity $c$. Therefore, in a highly conductive medium an RT method will fail if the picked traveltimes are used to reconstruct the $c$ or $\varepsilon$.

Figure 2 plots the electric field response of an impulsive line current source separated 30 m from the receiver; the top (bottom) figure is for a homogeneous medium with a conductivity of .001 (.01) S/m. The dielectric constant is $\varepsilon_r = 3.0$. In a homogeneous medium with $\sigma = 0$, the first arrival is associated with the impulse at $t = R/c$ as shown by the arrows. The effect of the more
Figure 2. Impulse response of the damped wave equation in a whole space for conductivities of (a) .001 S/m and (b) .01 S/m. The distance between the line current source and electric field receiver is 30m. Arrow indicates the radar wave first arrival associated with $G_{dw}$. 
conductive whole space is to delay and broaden this impulse so that the picked traveltime is related to both the \( \varepsilon \) and \( \sigma \) (Figure 2b). Figure 3 is the same as Figure 2 except a Ricker wavelet is used as the source wavelet and \( \varepsilon = 5 \varepsilon_0 \).

Figures 2 and 3 demonstrate that an increase in conductivity not only increases the attenuation coefficient (transmission loss), but also delays the traveltime of the first arrival, i.e., the picked traveltime data are less sensitive to \( \varepsilon \) as \( \sigma \) increases. Because a ray tracing method only accounts for the wave propagation effect and neglects diffusion effects, then an RT method will not converge to the true dielectric distribution. However, the WT method does account for both wave propagation and diffusion effects by matching the finite-difference calculated radargrams to the observed radargrams.

Physical Interpretation of Gradient as Wavepaths

The gradient function in equation (7) is similar to the gradient derived in Luo and Schuster (1990). It can be interpreted as follows: the traveltime quasi-residuals at receivers \( x_r \) can be thought of as the source time history \( j_r(x_r, t; x_s) \), where

\[
j_r(x_r, t; x_s) = -\frac{1}{\mu F} \frac{d(x_r, t+\Delta T^*; x_s) \Delta T^*(x_r, x_s)}{d(x_r, t; x_s)}\]

of a line current acting at \( x_r \). The traveltime residuals weight the observed radargrams, which are then back-projected in reverse time history along "wavepaths" from \( x_r \) to \( x \). The back-projected field is cross-correlated with the \( \vec{E}_y \) to give the gradient.

The gradient function can also be interpreted as a wavepath. Unlike a pencil thin ray, a wavepath can be thought of as a "fat" ray (Woodward, 1992) that defines those parts of the model which can act as scatterers contributing energy
Figure 3. Synthetic radargram recorded in a whole space at a distance of 30m from the line current source. The dielectric constant of the whole space is 5.0 and the conductivity values are (a) .001 S/m and (b) .01 S/m.
to the first arrival. For the special case of an impulsive source, the wavepath reduces to a pencil thin ray so that the traveltime residual is distributed along a raypath.

To illustrate the geometric interpretation of wavepaths, the gradient is computed for a source-receiver pair in the Figure 4 cylinder model. We fix the dielectric distribution (12\(\varepsilon_0\) for the background and 6\(\varepsilon_0\) for the cylinder) and evaluate the gradient term in equation (7) for the two cases of low (\(\sigma = .001\) S/m) and highly conductive (\(\sigma = .01\) S/m) rocks. The results are depicted in Figures 5a (\(\sigma = .001\) S/m) and 5b (\(\sigma = .01\) S/m). Figure 5a shows that the wavepath or gradient contours for the weakly conductive \(\sigma = .001\) S/m case are concentrated between and within the high velocity cylinder. Unlike a ray tracing method that updates the velocity model along a single pencil thin raypath, the WT method updates the velocity model along a "fat" ray or wavepath (Luo, 1991; Woodward, 1992). As conductivity increases (\(\sigma = .01\) S/m), the gradient distribution in Figure 5b becomes dominated by the diffusion phenomena, resulting in large attenuation and broadened wavepath widths. The consequence is less velocity resolution because the first arrival energy is influenced by a larger region in the model (see Figures 5c and 5d). Thus, more iterations are needed for highly conductive rocks with the likelihood of divergence for rocks with conductivities greater than .01 S/m. In other words the first arrival traveltimes become insensitive to the dielectric distribution when rock conductivity is large and diffusion effects are significant.
Figure 4. Cylinder model geometry. (a) Dielectric cylinder model, where $\varepsilon_r = \varepsilon/\varepsilon_0$. (b) The vertical cross section through the middle of (a). (c) The true velocity model calculated from $c = 1/\sqrt{\mu_0\varepsilon}$. (d) The vertical cross section through the middle of (c).
Figure 5. WT gradient for a source-receiver pair in the Figure 4 cylinder model with a homogeneous conductivity of (a) .001 S/m and (b) .01 S/m. Figures (c) and (d) show the values of the gradient function along a vertical cross section of Figures (a) and (b), respectively.
NUMERICAL RESULTS

This section presents the results of applying the WT and ray tracing travel-time inversion (RT) methods to both synthetic and real crosswell radar data. We use a parameterized ray tracing code (Sun, 1991) and a simultaneous iterative reconstruction technique (SIRT) (Lager and Lytle, 1977) for the tomography algorithm. The synthetic tests were designed to test the effectiveness of the WT method on noisy data and on rocks with a range of conductivities. The first two synthetic tests use the cylinder model and a complex 2-D fault model, and the third test uses a three-layer model. The final test uses crosshole radar data from the Stripa Project in Sweden.

In the synthetic data tests, 2-D radar data were generated by a 2-4 staggered grid finite-difference scheme applied to Maxwell's equations (Yee, 1966). The first break arrival times were automatically picked from these synthetic radagrams. A transient line source current is used and details are given in appendix B. Typical grid sizes were 101×101 (excluding an annulus 50 points wide that is used as an absorbing region) and for forward modeling of 1000 time steps about 2 minutes of CPU time were used on a Stardent 2000 computer.

Synthetic Data Tests

Conductivity variation test. These tests assess the effectiveness of the WT and RT methods on rocks where the conductivity varies between .001 S/m and .01 S/m. The synthetic cylinder model used here has exactly the same geometry as that used in the interpretation section (see Figure 4). In all cases, the model size is 30m×30m (101×101 grid points), the source function is a 2-D Ricker wavelet with a dominant frequency of 20 MHz, and there are 20 (150) sources
(receivers) evenly distributed along the model edges. The background conductivity is taken to be the same value as the cylinder conductivity.

Figure 6a depicts the WT tomogram after 4 iterations for the cylinder with a conductivity of $\sigma = 0.001$ S/m. The traveltime residuals are plotted in Figure 6b. After 4 iterations, the WT tomogram provides an accurate representation of the true model. In contrast, the RT tomogram in Figure 7 provides a poor reconstruction of the true cylinder (Figure 7a) and the traveltime residuals do not decrease after 10 iterations (see Figure 7b). Apparently, the rock conductivity introduces a diffusion-like time shift to the radargrams that is unaccounted for in the RT inversion. The RT method incorrectly accounts for the diffusion time lag by introducing slower velocities in the tomogram. This contrasts with the WT method that automatically accounts for the diffusion time shift by incorporating conductivity into the model. Of course, this assumes that the true conductivity is known, which is the case for this cylinder test.

Figures 8 and 9 are the same as in Figures 6 and 7 respectively, except that the conductivity of the medium is increased to 0.01 S/m. Here, the WT tomogram (Figure 8a) is reasonably accurate, but not as accurate as that in Figure 6a. This is because the calculation of the WT traveltime residual is now significantly influenced by both $\sigma$ and $\varepsilon$. In contrast, the RT tomogram shows a maximum cylinder velocity $c(x)$ that is 1/3 of its true value. Therefore, as conductivity increases the RT tomogram fails while the WT method still provides useful results.

Now we simulate radar data associated with the complex 2-D fault model in Figure 10a. In this case the model size is 30m×60m (61×121 grid points), there are 21 (21) sources (receivers) evenly distributed along the left (right) vertical well, and the source wavelet frequency is peaked at 10 MHz. The WT tomogram (after 13 iterations) and the RT tomogram (after 15 iterations) for this model are depicted in Figures 10c and 10d respectively, where Figure 10b is the starting
Figure 6. WT reconstruction of the cylinder model with a homogeneous conductivity of $\sigma = .001 \text{ S/m}$. (a) WT tomogram (upper left figure) after 4 iterations and the cross sections (upper right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals as a function of iteration number.
Figure 7. RT reconstruction of the cylinder model with a homogeneous conductivity of $\sigma = 0.001 \text{ S/m}$. (a) RT tomogram (upper left figure) after 10 iterations and the cross sections (upper right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals as a function of iteration number.
WT Tomogram

After 5 Iterations

(a)

Velocity (m/μs)

Depth (m)

(b)

Traveltime Residual (ns)

Number of Iterations

Figure 8. WT reconstruction of the cylinder model with a homogeneous conductivity of $\sigma = .01$ S/m. (a) WT tomogram (upper left figure) after 5 iterations and the cross sections (upper right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals as a function of iteration number.
Figure 9. RT reconstruction of the cylinder model with a homogeneous conductivity of $\sigma = .01$ S/m. (a) RT tomogram (upper left figure) after 10 iterations and the cross sections (upper right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals as a function of iteration number.
Figure 10. Reconstructions of the 2-D fault model. (a) Fault model. (b) Initial guess. (c) WT tomogram after 13 iterations. (d) RT tomogram after 15 iterations.
model. For this complex dielectric distribution with a $\varepsilon_r$ range of 4.3–20., the WT tomogram delineates the correct dip and velocity of the dipping layer and roughly outlines the features of the top section. In comparison, the RT tomogram images the top section somewhat well but it distorts the shape of the dipping layer.

Finally, we test the sensitivity of the WT method to an inaccurate estimate of conductivity for the cylinder model. The WT procedures are repeated except an incorrect conductivity is used for the starting model. For the medium with .001 S/m conductivity, an incorrect conductivity $\sigma = 0$ is used in the starting model. The reconstructed tomogram (Figure 11) still resembles the actual model. For the highly conductive cylinder model with $\sigma = .01$ S/m, we first use an incorrect conductivity $\sigma = 0$ for the initial model, and Figure 12a shows that the WT tomograms diverge from the true model and the traveltime residuals increase (Figure 12b) with the number of iterations. This is because the estimated conductivity model ($\sigma = 0$) is too far away from the true value (.01 S/m). Repeating this inversion procedure using $\sigma = .005$ S/m for the homogeneous starting model, i.e., with a 50% inaccurate estimate of conductivity, shows (Figure 13) that the reconstructed tomogram converges to the true model. However, the result is not as accurate as that in Figure 8a where the correct $\sigma$ is used in the starting model. This suggests that for large conductivities ($\geq .01$ S/m), the diffusion effects cannot be neglected. It also suggests that the WT method is not very sensitive to a moderately inaccurate ($\approx 50\%$ error) estimate of background conductivity.

**Random noise test.** Considering a more realistic situation, we apply both the WT and RT methods to a set of synthetic data associated with a three-layer model. Here 5% random noise is added to the traveltime data. The model geometry is depicted in Figure 14, where the model size is 50m×75m (101×151 grid points), and the conductivities for the three layers from top to bottom are
Figure 11. WT reconstruction of the cylinder model with $\sigma = .001$ S/m using an incorrect estimate of conductivity $\sigma = 0$. (a) WT tomogram after 6 iterations (left figure) and cross sections (right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) The traveltime residual plotted against the iteration number.
Figure 12. WT reconstruction of the cylinder model with $\sigma = .01$ S/m using an incorrect estimate of conductivity $\sigma = 0$. (a) WT tomogram after 1 iteration (left figure) and cross sections (right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals plotted against the iteration number.
Figure 13. WT reconstruction of the cylinder model with $\sigma = .01$ S/m using an incorrect estimate of conductivity $\sigma = .005$. (a) WT tomogram after 5 iterations (left figure) and cross sections (right figure) through the middle of the tomogram (dashed line) and the actual model (solid line). (b) Traveltime residuals as a function of iteration number.
Figure 14. Reconstructions of the three-layer model. The upper (lower) row shows the WT (RT) tomograms for different iterations. Five percent of random noise is added to the synthetic traveltimes data.
geometry is depicted in Figure 14, where the model size is 50m×75m (101×151
grid points), and the conductivities for the three layers from top to bottom are
.001, .0001 and .0015 S/m, respectively. The source function is a Ricker wavelet
with a dominant frequency of 10 MHz, there are 16 sources evenly spaced on
the left edge of the model and 25 receivers evenly distributed along the surface
and the right vertical well. Figure 14 shows both the WT and RT tomograms.
Both methods appear to be insensitive to a realistic random noise level in the
taveltime picks.

Real Data Test

The actual radar data used in this test are obtained from two boreholes near
the Stripa mine (rock conductivity is about .001 S/m) in Sweden. Figure 15 depic-
ts the 2-D geometry of the boreholes. There are 29 sources and 29 receivers at
the upper and lower wells, respectively. Here we only use 26 sources and 29 rec-
ivers because the first three sources are in a water saturated zone. The sources
are magnetic dipoles with a peak frequency of 60 MHz. For the WT reconstruc-
tion, we use the picked first arrival traveltimes and synthetic Ricker wavelets in
the inversion procedure, i.e., the real radargrams are replaced by Ricker wavelets
shifted to the observed traveltimes. Due to memory limitations in the computer,
the Ricker wavelet in the computed radargrams has a peak frequency of 15 MHz
compared to the actual source frequency of 60 MHz.

Since the traveltimes of the electromagnetic waves are on the order of a few
micro-seconds, imprecise knowledge of the source starting time will adversely
influence the inversion. Therefore, preprocessing is needed to find the correct
starting time. Plotting the picked traveltime as a function of source-receiver dis-
tance (see Figure 16), and extrapolating to zero offset gives an estimate of
\( t_0 = 91.74 \) nano-seconds as the starting time. Figure 17a depicts a common
Figure 15. Two-dimensional geometry of the two boreholes in the Stripa mine, Sweden.
Figure 16. A plot of picked traveltimes from the real radargrams as a function of source-receiver distance. The starting time of the source is found at the zero offset position.
source point gather of the real radar data (vertical component of electric field) after the starting time correction, and Figure 17b shows the common source point gather of the synthetic radargrams generated from the WT tomogram in Figure 18b. The synthetic first arrival traveltimes are nearly the same as the observed traveltimes, although the waveforms of the synthetic radargrams are broader because of the narrower frequency band in the synthetic wavelets. For traveltime inversion, however, using a 15 MHz source wavelet may give results just as good as those from a 60 MHz source.

In Figure 18, (a) depicts the initial guess model for the two methods, (b) shows the WT tomogram after three iterations, (c) gives the ray tracing tomogram after 10 iterations and Figure 19 is the traveltime residual plotted as a function of iteration number. These two methods show similar results, i.e., the interwell region has a very small (less than 10%) velocity contrast. Since the rock conductivity is small, diffusion effects are negligible so that the RT method also works just as well as the WT method.

Unfortunately, we do not know the exact dielectric distribution between these boreholes, so that the accuracy of these tomograms is unknown. Because of the consistency between the WT and the RT tomograms we conclude that the WT method works as well as RT tomography for this real data set.
Figure 17. A common source point gather from: (a) real radar data after applying the starting time correction and (b) synthetic radargrams for the same source point. Synthetic data are generated using finite-difference solutions to Maxwell's equations and the velocity distribution reconstructed by the WT method.
Figure 18. Reconstructions of the real radar data. (a) Initial guess for both WT and RT tomograms. (b) WT result after 3 iterations. (c) RT tomogram after 10 iterations.
Figure 19. Traveltime residuals versus iteration numbers for the real data inversion. Stars and circles represent the WT and RT residuals, respectively.
DISCUSSION AND CONCLUSION

The wave equation traveltime inversion (WT) method is adapted to the reconstruction of the dielectric distribution. This method uses a finite-difference solution to the 2-D Maxwell’s equations in the space time domain, and so, unlike an RT method, it takes into account both the diffusion and scattering effects in the travel times. This is the most important advantage of the WT method over an RT method for dielectric parameter reconstruction. In addition, the WT method accounts for diffraction and multipathing effects that are not honored by an RT method. Synthetic data tests show that the WT method is more suitable than an RT method in the reconstruction of the dielectric distribution (electromagnetic wave velocity), especially for conductive media with \( \sigma > .002 \) S/m. Numerical tests suggest that the WT method can also tolerate an inaccurate estimate of conductivity as a priori information and is insensitive to 5% realistic random noise in the traveltime data.

For the real data set, the WT method gives a result similar to that of the RT method. Both methods show almost a constant dielectric value between the two boreholes. In this case, well log data show that the rock conductivity is small (\( \sigma = .001 \) S/m) so that both the WT method and the RT method are about equally effective.

The major disadvantage of the WT method relative to RT inversion is an order of magnitude increase in CPU time. This cost can be decreased by using an implicit finite difference method and in the future may not be a problem with the introduction of gigaflop workstations. A future task is to reconstruct both the
dielectric constant and conductivity distributions from the traveltimes and amplitudes of radargrams.
APPENDIX A

STAGGERED GRID FINITE-DIFFERENCE SCHEME

A 2-4 staggered grid finite-difference scheme (Yee, 1966; Levander, 1988) is used to numerically solve the 2-D Maxwell’s equations,

\[
\begin{align*}
\mu \frac{\partial H_x}{\partial t} &= \frac{\partial E_y}{\partial z}, \\
\mu \frac{\partial H_z}{\partial t} &= -\frac{\partial E_y}{\partial x}, \\
\frac{\partial E_y}{\partial t} &= \frac{1}{\varepsilon} \left( -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z} - \sigma E_y - j_y \right).
\end{align*}
\]

(A-1)

Here the source function \( j_y \) is given by a Ricker wavelet and the initial conditions are given as

\[
\begin{align*}
E_y(x, t; x_0) &= 0 \quad \text{for } t \leq 0, \\
H_x(x, t; x_0) &= 0 \quad \text{for } t \leq 0, \\
H_z(x, t; x_0) &= 0 \quad \text{for } t \leq 0.
\end{align*}
\]

Discretization of equation (A-1) is shown below, where a 4-point central differencing scheme is used to approximate the spatial derivatives and a 2-point central difference is used for the time derivatives.
The discretized Maxwell’s equations are written as

\[
\begin{align*}
\frac{HX_{i,j}^{t+1} - HX_{i,j}^t}{\mu \Delta t} &= c_1(\Delta t^{t+1/2} EY_{i,j-1}^{t+1/2} - EY_{i,j}^t) + c_2(\Delta t^{t+1/2} EY_{i,j-2}^{t+1/2} - EY_{i,j+1}^t), \\
\frac{HZ_{i,j}^{t+1} - HZ_{i,j}^t}{\mu \Delta t} &= c_1(\Delta t^{t+1/2} EY_{i+1,j}^{t+1/2} - EY_{i,j}^t) + c_2(\Delta t^{t+1/2} EY_{i+2,j}^{t+1/2} - EY_{i-1,j}^t), \\
\frac{\varepsilon_{ij}^{t+1/2} EY_{i,j}^{t+1/2} - EY_{i,j}^{t+1/2}}{\Delta t} &= c_1(HX_{i,j}^{t+1/2} - HX_{i,j+1}^t) + c_2(HX_{i,j-1}^{t+1/2} - HX_{i,j+1}^t) \\
&+ c_1(HZ_{i,j}^{t+1/2} - HZ_{i-1,j}^t) + c_2(HZ_{i+1,j}^{t+1/2} - HZ_{i-2,j}^t), \\
&- \Delta t \sigma_{ij}(EY_{i,j}^{t+1/2} + EY_{i,j}^{t+1/2})/2 - J_{i,j}^t,
\end{align*}
\]

where \( \Delta t \) is the time interval, \( \Delta x = \Delta z \) is the spatial grid size, \( c_1 = 9/8 \) and \( c_2 = -1/24 \), and \( J \) is the source function. Note that the superscript \( t \) denotes the time step and that the electric and magnetic fields are staggered in time relative to one another.

Rearranging the above equations gives an explicit time stepping equation

\[
\begin{align*}
HZ_{i,j}^{t+1} &= HZ_{i,j}^t - \frac{\Delta t}{\mu \Delta x}[c_1(\Delta t^{t+1/2} EY_{i+1,j}^{t+1/2} - EY_{i,j}^t) + c_2(\Delta t^{t+1/2} EY_{i+2,j}^{t+1/2} - EY_{i-1,j}^t)], \\
HX_{i,j}^{t+1} &= HX_{i,j}^t + \frac{\Delta t}{\mu \Delta x}[c_1(\Delta t^{t+1/2} EY_{i,j-1}^{t+1/2} - EY_{i,j}^t) + c_2(\Delta t^{t+1/2} EY_{i+2,j}^{t+1/2} - EY_{i,j+1}^t)], \\
EY_{i,j}^{t+1/2} &= \frac{2\varepsilon_{ij} - \sigma_{ij} \Delta t}{2\varepsilon_{ij} + \sigma_{ij} \Delta t} EY_{i,j}^{t-1/2} - \frac{2\Delta t}{2\varepsilon_{ij} + \sigma_{ij} \Delta t} J_{i,j}^t \\
&+ \frac{2\Delta t}{2\varepsilon_{ij} + \sigma_{ij} \Delta t} \left[c_1(HX_{i,j}^t - HX_{i,j+1}^t) + c_2(HX_{i,j-1}^t - HX_{i,j+1}^t) \\
&- c_1(HZ_{i,j}^t - HZ_{i-1,j}^t) - c_2(HZ_{i+1,j}^t - HZ_{i-1,j}^t) \right].
\end{align*}
\]
These finite-difference formulae are used to implement the forward modeling. The CFL stability condition for these equations is given by $\frac{\Delta t}{\Delta x} c_{\text{max}} < \frac{1}{\sqrt{2}}$ (Oristaglio and Hohmann, 1984; Levander, 1988) and the numerical dispersion criteria is $\lambda_{\text{min}} > 5\Delta x$, where $\lambda_{\text{min}}$ is the minimum wavelength associated with the source function.
APPENDIX B

FRECHET DERIVATIVE

Traveltome Inversion

To obtain the Fréchet derivative of the electric field with respect to a perturbation in the electromagnetic velocity, consider the second order Maxwell’s equation,

\[ \nabla^2 E_y(x, t) - \mu \sigma \frac{\partial E_y(x, t)}{\partial t} - \frac{1}{c^2(x)} \frac{\partial^2 E_y(x, t)}{\partial t^2} = \mu \frac{\partial j_y}{\partial t}. \quad (B-1) \]

To solve for \( E_y \) in the above equation, we introduce the Green’s function \( G \) and adjoint Green’s function \( G^* \) defined by \( G(x_0, -t_0; x, -t) = G^*(x_0, t_0; x, t) \) (Morse and Feshbach, 1953). \( G \) and \( G^* \) satisfy

\[ (\nabla^2 - \mu \sigma \frac{\partial}{\partial t} - \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2}) G(x, t; x_0, t_0) = \delta(x-x_0) \delta(t-t_0), \quad (B-2a) \]

and

\[ (\nabla^2 + \mu \sigma \frac{\partial}{\partial t} - \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2}) G^*(x, t; x_1, t_1) = \delta(x-x_1) \delta(t-t_1), \quad (B-2b) \]

with the initial conditions

\[ G(x, t; x_0, t_0) = 0 \quad \frac{\partial G(x, t; x_0, t_0)}{\partial t} = 0 \quad \text{for} \ t \leq t_0, \quad (B-3a) \]

and

\[ G^*(x, t; x_1, t_1) = 0 \quad \frac{\partial G^*(x, t; x_1, t_1)}{\partial t} = 0 \quad \text{for} \ t \geq t_1. \quad (B-3b) \]

From the reciprocity principle, \( G(x, t; x_0, t_0) = G^*(x_0, t_0; x, t) \), so that \( G \) and
$G^*$ can be written as functions of $t_1$ and $t_0$ respectively.

\[(\nabla^2_1 + \mu \sigma \frac{\partial}{\partial t_1} - \frac{1}{c^2(x_1)} \frac{\partial^2}{\partial t_1^2})G(x, t; x_1, t_1) = \delta(x-x_1) \delta(t-t_1), \quad (B-4a)\]

and

\[(\nabla^2_0 - \mu \sigma \frac{\partial}{\partial t_0} - \frac{1}{c^2(x_0)} \frac{\partial^2}{\partial t_0^2})G^*(x, t; x_0, t_0) = \delta(x-x_0) \delta(t-t_0). \quad (B-4b)\]

We also need to consider the equation

\[(\nabla^2_1 - \mu \sigma \frac{\partial}{\partial t_1} - \frac{1}{c^2(x_1)} \frac{\partial^2}{\partial t_1^2})E_y(x_1, t_1) = \mu \frac{\partial j_y(x_1, t_1)}{\partial t_1}. \quad (B-5)\]

Multiplying equation (B-5) by $G$ and equation (B-4a) by $E_y$, subtracting the two equations, and then integrating over the relevant volume in space $x_1$ and over time from 0 to $t^*$, we get,

\[E_y(x, t) = \int_0^{t^*} dt_1 \int dx_1 \mu \frac{\partial j_y(x_1, t_1)}{\partial t_1} G(x, t; x_1, t_1) \]

\[+ \sigma \mu \int dx_1 \left[ E_y(x_1, t_1) G(x, t; x_1, t_1) \right]_{t_1=0}^{t_1=\infty} \]

\[+ \int dx_1 \frac{1}{c^2(x_1)} \left[ G(x, t; x_1, t_1) \dot{E}_y(x_1, t_1) - \dot{E}_y(x_1, t_1) G(x, t; x_1, t_1) \right]_{t_1=0}^{t_1=\infty} \]

\[+ \int_0^{t^*} dt_1 \int_{s_1} [G(x, t; x_1, t_1) \nabla_1 E_y(x_1, t_1) - \dot{E}_y(x_1, t_1) \nabla_1 G(x, t; x_1, t_1)] ds_1. \quad (B-6)\]

Because the initial conditions for $E_y$ and $\partial E_y/\partial t$ should satisfy homogeneous initial conditions

\[E_y(x, 0) = 0 \quad \dot{E}_y(x, 0) = 0 \quad \text{for } t \leq 0, \]

then the second and the third integrands of equation (B-6) should vanish, and the last integrand is the effect of the boundary conditions. If adjoint boundary
conditions are chosen such that

\[
\int_{s_1} \left( G(x, t; x_1, t_1) \nabla_1 E_y(x_1, t_1) - E_y(x_1, t_1) \nabla_1 G(x, t; x_1, t_1) \right) ds_1 = 0, \quad (B-7)
\]

the effect of the boundary conditions can be neglected over an infinite space.

For large \( R \) and zero conductivity, \( G \) is asymptotically proportional to \( 1/R \)
\((R = |x - x_1| \to \infty) \) and \( \nabla_1 E_y(x_1, t_1) \) is asymptotically proportional to \( 1/R^2 \),
\( G(x, t; x_1, t_1) \to 0 \) and \( E_y(x_1, t_1) \to 0 \) as \( R \to \infty \). Hence, the far field asymptotic solution of equation (B-5) becomes

\[
E_y(x, t) = \int_0^{+\infty} dt_1 \int \mu \frac{\partial j_y(x_1, t_1)}{\partial t_1} G(x, t; x_1, t_1) dx_1. \quad (B-8)
\]

A small perturbation of velocity \( c(x) \to c(x) + \delta c(x) \) will cause a field perturbation \( E_y(x, t; x_s) + \delta E_y(x, t; x_s) \) for a source at \( x_s \). This \( \delta E_y \) satisfies the second order Maxwell’s equation

\[
(\nabla_1 - \mu \sigma \frac{\partial}{\partial t_1} - \frac{1}{[c(x_1) + \delta c(x_1)]^2} \frac{\partial^2}{\partial t_1^2} ) (E_y + \delta E_y) = \mu \frac{\partial j_y}{\partial t_1}, \quad (B-9)
\]

and the initial conditions

\[
E_y(x_1, 0; x_s) + \delta E_y(x_1, 0; x_s) = 0 \quad \dot{E}_y(x_1, 0; x_s) + \delta \dot{E}_y(x_1, 0; x_s) = 0.
\]

Using

\[
\frac{1}{[c(x_1) + \delta c(x_1)]^2} \approx \frac{1}{c(x_1)^2} - \frac{2 \delta c(x_1)}{c(x_1)^3},
\]

and subtracting equation (B-5) from (B-9) gives

\[
(\nabla^2 - \mu \sigma \frac{\partial}{\partial t} \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} ) \delta E_y(x, t) = -\frac{2}{c^3(x)} \delta c(x) \frac{\partial^2 E_y(x, t)}{\partial t^2}. \quad (B-10)
\]

We assume that \( \delta E_y \) also satisfies the adjoint boundary conditions (equation (B-
7), so that the solution of equation (B-10) can be represented by

\[
\delta E_y(x, t) = -\int_0^t dt_1 \int dx_1 \frac{2c(x_1)}{c^3(x_1)} \frac{\partial^2 E_y(x_1, t_1)}{\partial t_1^2} G(x, t; x_1, t_1).
\]  

(B-11)

Assuming that the perturbation is at a single pixel, i.e.,

\[
\delta c(x_1) = \delta(x' - x_1) \Delta c(x_1),
\]

and substituting this relation into equation (B-11) gives

\[
\delta E_y(x, t; x_s) = -\int_0^t dt_1 \int dx_1 \frac{2\delta(x' - x_1)\Delta c(x_1)}{c^3(x_1)} \cdot E_y(x_1, t_1; x_s) G(x, t; x_1, t_1) dx_1
\]

\[
= -\frac{2\Delta c(x')}{c^3(x')} \int_0^t dt_1 \cdot E_y(x', t_1; x_s) G(x, t; x', t_1),
\]

Dividing by \(\Delta c(x')\) and taking the limit of \(\Delta c \to 0\) gives

\[
\frac{\partial E_y(x, t; x_s)}{\partial c(x')} = \lim_{\Delta c \to 0} \frac{\delta E_y(x, t; x_s)}{\Delta c(x')}
\]

\[
= \frac{-2}{c^3(x')} \int_0^t dt_1 \cdot E_y(x', t_1; x_s) G(x, t; x', t_1). \quad \text{(B-12)}
\]

which is the expression for the Fréchet derivative.

**Traveltime and Waveform Inversion**

Equation (B-12) can be used to find \(\sigma(x)\) by minimizing the waveform misfit function \(e_\sigma\), i.e.,

\[
e_\sigma = \frac{1}{2} \sum_r \sum_j \int \Delta E_y^2 dt,
\]

(B-13)

where \(\Delta E_y\) is the difference between the calculated first arrivals in the electrograms and the observed electrograms. In this case, the gradient of \(e_\sigma\) is given by
\[
\frac{\partial e_\sigma}{\partial \sigma(x)} = \sum_r \sum_s \int \Delta E_y \frac{\partial E_y}{\partial \sigma(x)} \, dt,
\] (B-14)

where the Fréchet derivative \( \frac{\partial E_y}{\partial \sigma(x)} \) is given by

\[
\frac{\partial E_y(x, t; x_s)}{\partial \sigma(x)} = \mu \int_0^{t^+} dt_0 \, \dot{E}_y(x, t_0; x_s) \, G(x_r, t; x, t_0).
\] (B-15)

The above expression can be derived in a manner similar to that used for equation (B-12). We now have a means for determining both the conductivity and electromagnetic velocity with the following gradient formula.

\[
c^{(k+1)} = c^{(k)} - \alpha^{(k)} \frac{\partial e}{\partial c},
\]
\[
\sigma^{(k+1)} = \sigma^{(k)} - \beta^{(k)} \frac{\partial e_\sigma}{\partial \sigma}.
\]

where \( \frac{\partial e_\sigma}{\partial \sigma} \) is given in equation (B-14) and \( \frac{\partial e}{\partial c} \) is given in equation (7).
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From the calculation of the cross-correlation function $f$ we can find the optimal lag $\Delta T^*$ that provides the maximum value of $f$ and then use equation (7) to get the gradient $\gamma(x)$.

(4) The velocity update formula is given as (Luo and Schuster, 1990)

$$c^{(k+1)}(x) = c^{(k)}(x) - \alpha^{(k)} \gamma(x),$$

(10) where the $k$ index denotes the $k$th iteration, $\alpha^{(k)}$ is the step length and the velocities are iteratively updated by equation (10). For this nonlinear problem we use the steepest descent method to reconstruct the velocity distribution, but other gradient formula such as conjugate gradients can also be used.